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KILLING FIELDS IN COMPACT LORENTZ 3-MANIFOLDS

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Abstract

Here we classify flows on compact 3-manifolds that preserve smooth Lorentz metrics.

1. Introduction

The geodesic and horocyclic flows on the unit tangent bundle of a hyperbolic surface are well known by their beautiful, but very different properties. Nervertheless, these two flows with antagonistic dynamics are unified by the Lorentz geometry. By this, we mean that both of them are Killing fields for Lorentz structures. The purpose of this paper is to show that Lorentz geometry not only unifies but also characterize them. That is, the nontrivial (i.e., nonequicontinous) Killing fields for Lorentz metrics in dimension three, are all "derived from" geodesic or horocyclic flows.

Algebraically, the unit tangent bundle of the 2–hyperbolic space is identified with the group $PSL(2,\mathbf{R})$. The fundamental group of a hyperbolic surface is thus identified with a discrete subgroup Γ in $PSL(2,\mathbf{R})$, and its unit tangent bundle with $\Gamma\backslash PSL(2,\mathbf{R})$. A one-parameter group $\{f^t\}$ in $PSL(2,\mathbf{R})$ determines on $\Gamma\backslash PSL(2,\mathbf{R})$ a right translation flow $\Gamma x \to \Gamma x f^t$. The geodesic (resp. horocyclic) flow corresponds to the hyperbolic (resp. parabolic) one-parameter group: $g^t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix}$ (resp. $h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$) . In fact any noncompact one-parameter of $PSL(2,\mathbf{R})$ is conjugate to $\{g^{\alpha t}\}$ or $\{h^{\alpha t}\}$ for some real α. If a one-parameter group is compact, it is conjugate to $\left\{\begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix}\right\}$.

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The Killing form on the Lie algebra of $PSL(2,\mathbf{R})$ determines a bi-invariant Lorentz metric. It thus passes to a Lorentz structure on the left quotients $\Gamma\backslash PSL(2,\mathbf{R})$, which is preserved by the right translation flows. The deformations of the Lorentz structure and those of the right translation flows were independently discovered by W. Goldman [6] and E. Ghys [4]. They are constructed in the following way. Observe first that the flow determined by $\{f^t\}$ is preserved by the group $G = PSL(2,\mathbf{R}) \times \mathbf{R}$, where the left factor acts by the left translation and the $\mathbf{R}$ factor by the right translation by f^t, i.e., the flow itself. Therefore in order to get a flow which looks like that determined by $\{f^t\}$, and in particular preserving the Lorentz structure, we just need a geometric structure modeled on $(G, PSL(2,\mathbf{R}))$. One may imagine that this does not produce new flows. E. Ghys was the first to see the contrary. For this let us call them Ghys flows (so also the geodesic and horocyclic flows are now trivial Ghys flows). Now a deformation Γ' of Γ in G is given by a homomorphism $c : \Gamma \longrightarrow R$, so that $\Gamma' = Graph\ c = \{(\gamma, c(\gamma)) \in PSL(2,\mathbf{R}) \times \mathbf{R}\}$. Thus an element $\gamma' = (\gamma, c(\gamma))$ acts by $x \longrightarrow \gamma x f^{-c(\gamma)}$.

Next, for cocompact Γ we know that small deformations of Γ are realised by deformations of the geometric structure, and so small cohomological classes in $Hom(\Gamma, \mathbf{R})$ generate Ghys flows.

Other trivial (in a dynamical sense) examples of isometric flows of Lorentz manifolds, that we shall call equicontinuous flows, are those which in fact preserve Riemannian metrics. They are easy to understand (see section 2). Our principal result, is that, a nontrivial isometric flow on a Lorentz compact 3–manifold is conjugate (as a flow) to the suspension of a hyperbolic linear automorphism or to a Ghys flow. More precisely, we have:

Theorem 1. *Let (M, ϕ^t) be a smooth flow preserving a smooth Lorentz structure on a compact 3-manifold M. Suppose that ϕ^t is not equicontinuous. Then up to a rescale of a constant multiple of the parameter t (that is replacing ϕ^t by $\phi^{\alpha t}$ for some constant α), and up to finite covers, the flow is smoothly isomorphic (as a flow) to one of the following :*

- *i) The suspension with return time 1 of a toral hyperbolic linear diffeomorphism.*
- *ii) A Ghys flow on the complete Lorentz space form M of constant curvature. Moreover there is $\Gamma \subset PSL(2,\mathbf{R})$ isomorphic to the fundamental group of a compact surface and a homomorphism (not necessarily small) $c : \Gamma \longrightarrow \mathbf{R}$, so that for f^t a hyperbolic or parabolic one-parameter group, the manifold M is the quotient*

of $PSL(2, \mathbf{R})$ *by* $\Gamma' = Graph(c) = \{(\gamma, c(\gamma))\}$ *acting as* $x \longrightarrow \gamma x f^{-c(\gamma)}$. *In particular* Γ' *acts freely properly discontinuously on* $PSL(2, \mathbf{R})$.

Let us now give some comments about Theorem 1.

1) By "up to finite covers", we mean by taking a quotient of a finite cover of our manifold. It is sometimes necessary to start by passing to a finite cover as in the case of the geodesic and horocyclic flows of orbifolds.

2) Note that the suspension flows may be considered in some sense as a "limit" of geodesic flows. Indeed, such flows are obtained in an algebraic way, as above, with the group SOL instead of $PSL(2, \mathbf{R})$, and at the Lie algebra level *sol* is a limit of algebras isomorphic to that of $PSL(2, \mathbf{R})$.

3) **Singularities.** Note that we do not assume the flows are nonsingular. But it follows from our result that this is the case if they are nonequicontinuous.

4) **Regularity.** To simplify, we assume here the metric C^∞. It then follows that the isometric flow itself is C^∞. In fact the proof of Theorem 1 uses the existence of the curvature tensor for the metric and second derivatives for the flow. Hence the metric and the flow must be C^2. Our proof also uses somewheres Sard's theorem applied to functions derived (algebraically) from the curvature. Thus, they must be C^3 (since the dimension is 3) and so the metric must be C^5. Nervertheless we may avoid this use of Sard's theorem and so the C^2 hypothesis for the metric and the flow are enough.

Note on the other hand that, by Kanai's construction in [10], a volume preserving Anosov flow on a 3-manifold preserves a C^0-Lorentz metric, which may be C^1 (e.g. the geodesic flows of negatively curved surfaces). Nevertheless, the metric should be C^∞, if it is just C^2, and thus the flow is as in Theorem 1 above (this is the Ghys classification of Anosov flows with smooth stable and unstable distributions (see §2)).

5) **Isometry groups.** One may easily deduce from our results (see Theorem 2 below) that if the isometry group of a compact Lorentz 3–manifold is not discrete (i.e., noncountable), then it has finitely many components, and further the identity component, if noncompact, is isomorphic to $\mathbf{R}$ or $PSL(2, \mathbf{R})$ (all this, up to finite covers). In fact, our motivation in studying isometric flows on compact Lorentz manifolds was an attempt to understand the isometry groups of such manifolds, following the point of view of Gromov's theory on rigid transformation groups [7] (although we do not use here results from this theory). For known results on this field, one usually works with some hypothesis deal-

ing with the algebraic structure of the isometry group (e.g., it contains $SL(2,\mathbf{R})$, [18], [7]), or with the topology of the manifold (for instance it is simply connected as in [3]), or with the geometry of the underlying manifold, as in [9] where author assumes it to be 3-dimensional and of constant curvature.

Let us now give more informations about the invariant Lorentz metric and the differentiable structure of the flow. These may be extracted from the proof of Theorem 1 or deduced from Theorem 1 itself.

Theorem 2.

1) In the suspension case, all the invariant Lorentz metrics are flat and obtained from an initial metric by multiplying along the direction of the flow and its orthogonal by some constants.

2) The Killing metric, up to a multiplicative constant, is the only Lorentz metric of constant (negative) curvature invariant by a Ghys flow. In particular, an isomorphism between two Ghys flows is an isometry between their geometric structures.

3) In the case of Ghys flows, the invariant metrics correspond to the left invariant metrics on $PSL(2,\mathbf{R})$, determined by the scalar products on the Lie algebra which are furthermore $Ad(f^t)$-invariant, and so in particular these metrics are locally homogenous. Up to a multiplicative factor, the space of such scalar products is 1-dimensional.

3.1) In the case of hyberbolic one-parameter groups, these metrics correspond to the multiplication of the Killing metric by different constants along the flow and its orthogonal .

3.2) In the case of parabolic one-parameter groups, the 1-dimensional space of the scalar products (up to multiplicative factors) which are $Ad(h^t)$-invariant gives only 3 isometry types of metrics on $PSL(2,\mathbf{R})$. Nevertheless in M, these metrics are not (globally) isometric unless M is homogenous.

4) In the case of suspension flows or Ghys flows with f^t hyperbolic, the flow is (everywhere) spacelike. In the case of Ghys flows, with f^t parabolic, the flow is (everywhere) lightlike.

Theorem 3. *Up to finite covers, a Ghys flow is smoothly orbit equivalent to a geodesic or horocyclic flow on a surface of constant curvature.*

2. Preliminaries–steps of the proofs

A Lorentz scalar product on a vector space is a nondegenerate symmetric bilinear form $<,>$ of signature $-+...+$, e.g $\mathbf{R}^{n+1}$ endowed with the quadratic form $-dx_0^2+dx_1^2+...+dx_n^2$. A vector u is called **spacelike**, **timelike** or **lightlike** respectively, according to that $<u,u>$ is $>$

0, < 0, or $= 0$. Sometimes (perhaps for physical reasons) a Lorentz scalar product is defined to have a signature $+ - \ldots -$. Nevertheless the types must not depend on the convention of the signature, and may be defined (when the dimension is at least 3) in the following way. The set of lightlike vectors is a cone, called the light cone. It separates the space into 3 connected components, with two of them opposite. A vector is timelike if it belongs to one of these opposite components and otherwice spacelike.

The essential difference between Lorentz and Euclidean scalar products is that in the Lorentz case, the orthogonal $u^{\perp}$ of a vector u may contains this vector itself. This exactly happens when the vector is lightlike. In this case the restriction of the scalar product to $u^{\perp}$ is positive (this is why we choose the convention $- + \ldots +$ instead of $+ - \ldots -$) but not definite, with null space $\mathbf{R}u$. In general u is timelike (resp. spacelike) if and only if the restriction of the Lorentz scalar product to $u^{\perp}$ is positive definite (resp. a Lorentz scalar product).

A Lorentz structure on a manifold M is a smooth field of Lorentz scalar products of its tangent spaces. Notions of types for vectors or vector fields tangent to M are defined as in the previous case.

As in the riemannian case, Lorentz metrics give rise to a Levi–Civita connection. That is a torsion free connection, for which the metric is parallel. Also, as in the riemannian case, the isometry group of a Lorentz structure is a Lie group acting smoothly on the manifold. However, even if M is compact, in contrast with the riemannian case, the isometry group may not be compact. We shall say, as usual, that a subgroup G of $Isom(M)$ is **equicontinuous** if its closure in the group of homeomorphisms of M is compact. In fact this closure lies in $Isom(M)$. Thus the closure is a compact Lie group acting smoothly on M. From our point of view here such G may be said trivial (although preserving a riemannian metric and a Lorentz metric together may be sometimes, restrictive). Given nonequicontinuous groups G, the compactness together with the invariant geometric structure on M generates a beautiful dynamics that we are trying to understand when $\dim M = 3$ in this paper. The connection permits to define (parametrized) geodesics, exactly as in the riemannian case. However the affine parameter for the geodesics is not so easy to interpret via a distance. Nevertheless, the geodesics may be directly defined as in the riemannian case as critical points of a (nonpositive) lagrangian associated to the metric.

A fundamental difference between riemannian and Lorentz manifolds is that, in contrast to the former ones, for the latter, compactness does not imply completeness (that is, the definition of geodesics for all time).

Examples. We refer to [16] for a complete exposition about Lorentz manifolds of constant curvature. The flat ones which are complete and simply connected are isometric to the Minkowski space $\mathbf{R}^{n,1}$. That is, $\mathbf{R}^{n+1}$ endowed with a constant Lorentz scalar product. Let us also recall that in dimension 3, up to a multiplicative constant, manifolds with constant negative curvature are locally isometric to the group $PSL(2, \mathbf{R})$ endowed with its Killing form. Other "interesting" examples of homogenous Lorentz spaces will appear in sections 14 and 16.

What we shall really prove in this article, is the following result which implies Theorem 1.

Theorem 0. *Let $(M, <,>)$ be a compact Lorentz 3–manifold and ϕ^t an isometric flow on it, which is not equicontinuous. Then exactly one of the following two possibilities can occur :*

i) The flow is (everywhere) spacelike and Anosov.
ii) The flow is (everywhere) lightlike and preserves a complete Lorentz metric of constant negative curvature on M.

Let us now deduce Theorem 1 from Theorem 0 (see also §13). In the case i), the stable and the unstable directions are just the orthogonal isotropic directions of the the flow. They are in particular smooth. Hence by [4], the flow is isomorphic to a suspension of a linear hyperbolic diffeomorphism on a torus or to a Ghys flow, with f^t hyperbolic.

In the case ii) we apply a result of Kulkarni and Raymond [11], which states that, up to finite covers (and a multiplicative constant), a compact complete Lorentz 3–manifold of constant negative curvature is diffeomorphic to the unit tangent bundle of a hyperbolic surface. This result also shows that the holonomy is conjugate to $\Gamma' = \{(\gamma, c(\gamma)),\ \gamma \in \Gamma\}$ where $\Gamma \subset PSL(2, \mathbf{R})$ is a surface group, and $c\ : \Gamma \longrightarrow PSL(2, \mathbf{R})$ is a homomorphism. We know that a lift of the lightlike isometric flow to the Lorentz space $PSL(2, \mathbf{R})$ centralizes Γ. This easily implies that Γ is conjugate to a subgroup of $PSL(2, \mathbf{R}) \times \{h^t,\ t \in \mathbf{R}\}$, where $\{h^t\}$ is the parabolic one-parameter group defined in the previous section.

Remarks.

1. For our purpose, the classification by E. Ghys of Anosov flows with smooth distributions may be replaced by an elemantary argument (see §16).

2. J. Mess [13] proved the completeness of compact Lorentz 3–manifolds of constant negative curvature. His proof is an adaptation of the deep Carrière's proof in the flat case [1] to the more geometrically complicated case of constant negative curvature. Our proof here when the manifold supports a non equicontinuous isometric flow is quite elementary.

Steps of the proof of Theorem 0. We start in §3 by showing some uniformity results for isometric flows of Lorentz metrics. That is, the equicontinuity at some point of the derivative of the flow, or even a subsequence, implies the global equicontinuity of the flow itself. This is just derived from the Lie group structure for the group of isometries. Next for Lorentz (or just pseudo–riemannian) metrics, even, a "codimension-one" equicontinuity implies global equicontinuity. From this we deduce that if our isometric flow (M, ϕ^t) is nonequicontinuous, then it is nowhere timelike. We further prove in §4 that if it is somewhere spacelike, then it is everywhere spacelike, and is thus of Anosov type.

The remaining case is then when the flow is (everywhere) lightlike. In fact all the sections from 5 to 15 deal with it. The length of the proof in contrast with the spacelike case may be understood by the absence of general methods or principles for non-hyperbolic dynamical systems. We mean by this that for example, completeness, or nullity for some invariant tensors, are formally derived from the hyperbolicity, but this requires more analysis in the non-hyperbolic case. So in §5, we define adapted basis in which the derivative cocycle has a nice unipotent form. This implies (§11) that an invariant quadratic form, is not necessarily trivial in the sense that it is proportional to the metric, but has a special form (with respect to the metric). In particular, up to a multiplicative constant, the space of such invariant forms is 1–dimensional. We apply this to the Ricci tensor. In dimension 3 (and also 2 of course) the Ricci curvature determines the curvature tensor and so also the sectional curvature. In particular if it is proportional to the metric, i.e., M is Einstein, then the metric is of constant curvature, and we are done. If not, we consider this Ricci tensor itself as a metric and consider its Ricci curvature. Of course, all these lie in our 1–dimensional space of invariant quadratic forms. We may hope that this process, which is a very elementary version of the so called Ricci flow (for riemannian manifolds), converges to giving a fixed point which is thus an invariant Einstein metric. This program works more or less as described above, but requires a lot of preparations between §6 and §10. A principal ingredient for this study was the 2–plane field orthogonal to the flow (it contains the flow itself since it is lightlike). It is integrable with totally geodesic leaves. We may then consider the restriction of the Levi–Civita connexion to the leaves. We prove that it is locally symmetric and so completely describe it...

3. Uniformity

Let $(M, <>)$ be a compact Lorentz manifold, endowed with an auxiliary riemannian norm, denoted by $| \quad |$.

Proposition 3.1. *Let f_i be a sequence of isometries of $(M, <>)$. If for some point $x, D_x f_i$ are bounded (i.e., $|D_x f_i| < c$ for some constant c), then the sequence f_i is (uniformly) equicontinuous (i.e., $|Df_i| < C$ for some constant C) .*

Proof. This follows from the construction of the Lie group structure of the isometry group of M. Indeed let G be this group (endowed with the uniform topology), and $R(M)$ be the bundle of linear frames of M. Fix r_x a such frame for $T_x M$, and consider the evaluation map

$$e_x : G \to R(M); \quad e_x(f) = D_x f(r_x).$$

Then by the construction of the Lie group structure of G, e_x is a proper embedding. Our condition ensures that the images $e_x(f_i)$ lie in a compact subset of $R(M)$. Hence the sequence f_i is in a compact subset of G.

We also have the following stronger statement which follows from the continuity of the evaluation map e_x (with respect to x and r_x) .

Proposition 3.2. *If a sequence of isometries f_i is such that $D_{x_i} f_i$ are bounded for some sequence x_i, then f_i is equicontinuous.*

Corollary 3.3. *Let ϕ^t be an isometric flow of $(M, <>)$. If for some point x, a subsequence ϕ^{t_i} is equicontinuous at x, then ϕ^t is equicontinuous.*

Proof. Let L be the closure of the one-parameter group ϕ^t in the isometry group G. This is a cylinder $\mathbf{T} \times \mathbf{R}^k$, where $\mathbf{T}$ is a torus, and ϕ^t is a dense one-parameter group inside (i.e., a dense geodesic in affine geometric terms). But this is possible exactly when $L = \mathbf{R}$ or $L = \mathbf{T}$. Our hypothesis and the proposition above imply that ϕ^{t_i} is equicontinuous and hence L is a torus.

Remark 3.4. The above facts extend to compact manifolds equiped with a structure of an affine connection (e. g. a pseudo-riemannian metric).

Corollary 3.5. *If an isometric flow ϕ^t is somewhere timelike, then it is equicontinuous.*

Proof. Let x be a point where $X(x) = \frac{\partial \phi^t(x)}{\partial t}$ is timelike. Since ϕ^t is volume preserving and the set of timelike points is open, we may assume x to be recurrent. Let $t_i \to \infty$ be such that $x_i = \phi^{t_i} x$ tends to x. For any timelike point y, we transform the Lorentz product into a positive scalar product, in a canonical way, by only changing the sign along X (thus for the new positive scalar product $X^\perp$ is still orthogonal to X and is endowed with the initial scalar product). The timelike condition exactly allows that transformation. Obviously the flow preserves the riemannian metric (defined only in an open subset of M). Now the equicontinuity of $D_x\phi^{t_i}$ follows by evaluating at a compact neighbourhood of x containing the x_i. Hence the above corollary gives the equicontinuity of ϕ^t.

Proposition 3.6. *Let f_i be a sequence of isometries of $(M, <,>)$. Assume that for some x and some hyperplane $H \subset T_xM$, the restriction of the D_xf_i to H is equicontinuous (that is there are constants c and C such that $c|u| \leq |D_xf_i(u)| \leq C|u|$, if $u \in H$, where $|\ \ |$ in an auxiliary norm). Then the sequence f_i is equicontinuous.*

Proof. By 3.1, we have to prove that the sequence is equicontinuous at x. Next, we transform the problem to a linear one, by composing the f_i with isometric identifications between $T_{f_i(x)}M$ and T_xM. This preserves the equicontinuity condition as M is compact. Therefore we think of the f_i as linear isometries of the Minkowski space $\mathbf{R}^{n,1}(n+1 = \dim M)$. By our equicontinuity hypothesis, we may assume that $f_i|H$ converge to a linear injection $f : H \to \mathbf{R}^{n,1}$. Let $a_1, \cdots, a_n$ be a basis of H and $b_i = f(a_i)$, which span an hyperplane H'.

If H is non degenerate, that is $\det(< a_i, a_j >_{ij}) \neq 0$, then the same is true for H' because $< b_i, b_j >=< a_i, a_j >$. We complete $\{a_1, \cdots, a_n\}$ to a basis of $\mathbf{R}^{n,1}$ by adding a unitary vector a_{n+1} orthogonal to H : $< a_{n+1}, a_{n+1} >= \pm 1$ and $< a_k, a_{n+1} >= 0$, for $i \leq n$. Note that this system of equations has exactly a_{n+1} and $-a_{n+1}$ as solutions. Let b_{n+1} be a vector associated in the same way to $\{b_1, \cdots, b_n\}$. We see that $f_i(a_{n+1})$ has exactly two possible limits, b_{n+1} or $-b_{n+1}$, and therefore f_i is equicontinuous in this case.

Assume now that H is degenerate, say $< a_1, a_k >= 0$ for $1 \leq k \leq n$. Define a_{n+1} by the following equations : $< a_{n+1}, a_1 >= 1, < a_{n+1}, a_k >= 0$ for $2 \leq k \leq n$, and $< a_{n+1}, a_{n+1} >= c$, where c is an arbitrary constant (for example 0). To solve this system consider P the 2-plane orthogonal to the subspace generated by $\{a_2, \cdots, a_n\}$. The metric is definite in this last subspace and so is Lorentzian in P. Hence the remaining two equations $< a_{n+1}, a_1 >= 1$ and $< a_{n+1}, a_{n+1} >= c$ have exactly one solution. Indeed the solution of the first one is a one-dimensional affine subspace, $\{v + ta_1, t \in \mathbf{R}\}$. Thus the second equation is $< v, v > +t < v, a_1 > +t^2 < a_1, a_1 >=< v, v > +t = c$, and therefore t is unique.

Now as in the nondegenerate case, we see that $f_i(a_{n+1})$ tends to the solution b_{n+1} (for the same constant c). Hence f_i is equicontinuous in every case.

4. The spacelike case

Proposition 4.1. *Let $(M, <>)$ be a compact Lorentz $3-$manifold and ϕ^t an isometric flow. Assume ϕ^t to be nonequicontinuous and spacelike at some point, that is for some point $x_0, < X(x_0), X(x_0) >$ is positive, where X is the infinitesimal generator of ϕ^t. Then ϕ^t is everywhere spacelike. In fact $< X, X >$ is constant and in particular X is nonsingular.*

Proof. Consider a small transversal τ to X at x_0. It has an holonomy invariant Lorentz metric because X is Killing and spacelike. The function $f(x) =< X(x), X(x) >$ is ϕ^t-invariant and so determines an holonomy invariant function on τ, also denoted by f. Assume it is nonconstant and choose τ so small that its levels determine a trivial foliation in τ. As a Lorentz $2-$manifold, τ has two isotropic foliations. We may assume that at least one of them, say $\mathcal{L}$, is transverse to the levels of f. This determines (but not uniquely) a coordinates system $\{(a, b)\}$ for τ, with the levels of a (resp. b) corresponding to $\mathcal{L}$ (resp. the levels of f).

We may assume that x_0 is recurrent and projects to $y_0 \in \tau$. Thus there are holonomy elements (Poincaré return maps) γ_i such that $\gamma_i(y_0) \to y_0$, as $i \to \infty$. But the holonomy respects each level of f, and also $\mathcal{L}$ (but globally). Thus each γ_i has (in its domain of definition) the form $\gamma_i(a, b) = (\theta_i(a), b)$. Hence $D_{y_0}\gamma_i$ are equicontinuous along the tangent space of the level of f containing y_0. This means in M that the derivates of the associated isometries ϕ^{t_i} are equicontinuous on the hyperplane, tangent to the level of f (defined on M) at x_0.

Therefore, from 3.6 and 3.3, ϕ^t is equicontinuous, which contradicts our hypothesis.

This means that f is locally constant in the set of spacelike points. Let U be the connected component of x_0 in this set. Then it is the same as the component of x_0 in the set $\{x / f(x) = f(x_0)\}$. Hence it is open and closed. This proves the proposition as we tacitly assume that M is connected.

Proposition 4.2. *Under the conditions of the proposition above the flow ϕ^t is Anosov.*

Proof. By 4.1, X is everywhere spacelike. By passing if necessary to a finite cover, we may assume that the two isotropic directions in $X^\perp$ are oriented by two smooth vector fields Y and Z. By ϕ^t-invariance of these isotropic directions, we may write $D_x\phi^t(Y(x)) = a(x, t)Y(\phi^t(x))$ and $D_x\phi^t(Z(x)) = b(x, t)Z(\phi^t x)$. We now prove that for any x, the orbit $\{D_x\phi^t(Y(x)), t \in \mathbf{R}\}$ is not bounded in TM. The contrary means $a(x, t) \leq a$, for some real a. By the volume preserving property, $a(x, t)b(x, t) = 1$. If $a(x, t)$ stays $\geq a' > 0$ for a sequence t_n tending to $+\infty$ or $-\infty$, then $D_x\phi^{t_n}$ is equicontinuous and so by 3.3, the flow itself is equicontinuous, which contradicts our hypothesis. It then follows that $a(x,t) \to 0$, when $t \to \pm\infty$. Thus by continuity of $t \to a(x,t)$ there are t_n and t'_n tending to $+\infty$ such that $a(x,-t_n) = a(x,t'_n)$. Applying the cocycle property for a at $x_n = \phi^{-t_n}x$, we get $a(x_n, t'_n+t_n) = a(x,t'_n)a(x_n,t_n)$. But $a(x_n,t_n)a(x,-t_n) = 1$, and hence $a(x_n, t_n + t'_n) = 1$

Hence $b(x_n, t_n + t'_n) = 1$, and consequently $D_{x_n}\phi^{t_n+t'_n}$ is equicontinuous. Proposition 3.2 implies then that ϕ^t is equicontinuous (since $t_n + t'_n \to \infty$), which contradicts our assumption. In the same way $b(x,t)$ is not bounded and hence the orbit of any nonzero vector in $X^\perp$ by the tangent flow $D\phi^t$ is not bounded. This means by definition that ϕ^t is quasi-Anosov [12]. But in dimension 3, or in general for volume preserving flows, quasi-Anosov implies Anosov.

5. The lightlike case - The derivative cocycle

By 3.5 and 4.1, a nonequicontinuous Killing field which is somewhere lightlike, is everywhere lightlike. We assume that it is nonsingular for the moment. Thus the orthogonal $X^\perp$ is a 2−plane field, containing X itself, and is obviously preserved by ϕ^t.

Adapted basis. We may assume $X^\perp$ to be orientable, and then choose a unitary vector field Y in $X^\perp$, so that $< X, Y >= 0$ and $< Y, Y >= 1$. Note that Y is by no means unique but once chosen, and determines an unique vector field Z (not lying in $X^\perp$) such that $< X, Z >= 1, < Y, Z >= 0$ and $< Z, Z >= 0$. To see this, observe that Y is spacelike and hence $Y^\perp$ is lorentzian. Thus $Y^\perp$ possesses a second isotropic direction other than that of X. Therefore Z is uniquely defined by the auxiliary equation $< X, Z >= 1$. We shall call an **adapted basis** a basis field $\{X, Y, Z\}$ like above. Such a basis is not unique, and we shall see later how to add locally some differential relations, together with the above algebraic ones, among the vector fields X, Y and Z.

Consider now in $\mathbf{R}^3$ endowed with its canonical basis $\{e_1, e_2, e_3\}$, the lorentzian scalar product $<>:< e_1, e_1 >=< e_1, e_2 >=< e_3, e_3 >= 0$ and $< e_2, e_2 >=< e_1, e_3 >= 1$.

For any x, let $i_x : T_xM \to \mathbf{R}^3$ be the isomorphism sending the frame $\{X(x), Y(x), Z(x)\}$ to the canonical basis $\{e_1, e_2, e_3\}$. By definition, this establishes an isometry between the lorentzian scalar products.

The derivative cocycle $c(t,x)$ of ϕ^t is the matrix of $D_x\phi^t$ with respect to the given basis, i.e., $c(t,x) = i_{\phi^t(x)}D_x\phi^t i_x^{-1}$. The fact that ϕ^t is isometric translates to that $c(t,x)$ respects the lorentzian scalar product of $\mathbf{R}^3$.

Proposition 5.1. *The derivative cocycle has the form :*

$$c(t,x) = \begin{pmatrix} 1 & T(t,x) & \frac{-T^2(t,x)}{2} \\ 0 & 1 & -T(t,x) \\ 0 & 0 & 1 \end{pmatrix},$$

where $T : \mathbf{R} \times M \to \mathbf{R}$ is an additive cocycle, which uniformly (in x) goes to ∞ when $t \to \pm\infty$.

Proof. Note that the fact that $D\phi^t$ preserves X and $X^\perp$, means that $c(t,x)$ preserves e_1 and the plane generated by e_1 and e_2. If $c(t,x)e_2 = \alpha e_2 + a e_1$, then we write $< \alpha e_2 + a e_1, \alpha e_2 + a e_1 >=< e_2, e_2 >= 1$, and get $\alpha = 1$. Since $\det(c(t,x)) = 1$, we have the unipotent form for $c(t,x)$. To obtain the formula, we now just write : $< c(t,x)e_2, c(t,x)e_3 > =< e_3, e_3 >= 0$. Next, since ϕ^t is supposed to be nonequicontinuous, from 3.2 it is seen that $T(t,x)$ goes uniformly to ∞ when $t \to \pm\infty$.

Remark 5.2. The cocycle property allows us to prove that for some positive constant $\alpha, T(t,x)/t \geq \alpha$. Furthermore, by the subadditive ergodic theorem, for almost every x, there is $\beta_x \geq \alpha$ such that $\lim_{t\to\infty} T(t,x)/t = \beta_x$. But we can see that only in the homogeneous case, one can have $T(t,x) = \beta t$, that is, $c(t,x)$ is a one-parameter group of matrices.

6. The asymptotic foliation

Proposition 6.1. *The $2-$plane field $X^\perp$ is integrable. The tangent foliation $\mathcal{F}$, called the asymptotic foliation of X, has its leaves geodesic, that is, a geodesic tangent somewhere to a leaf is everywhere tangent to it.*

Proof. The integrability follows from the fact that ϕ^t preserves $X^\perp$. Indeed this means that for Y' tangent to $X^\perp, [X, Y']$ belongs to $X^\perp$. But (locally) $X^\perp$ is spanned by X and any Y' transverse to X.

To see that the leaves of the generated asymptotic foliation $\mathcal{F}$ are geodesic, note the following.

Fact 6.2. *For an adapted basis $\{X, Y, Z\}$ we can assume that $[X, Y] = 0$ in a neighbourhood of a given fixed point.*

Proof. Indeed, we choose Y in a small τ transversal to X and set $Y(\phi^t(x)) = D_x\phi^t(Y(x))$ in a neighbourhood of τ, for $x \in \tau$. Note that this still satisfies the condition $Y \in X^\perp$ and $< Y, Y >= 1$ because ϕ^t is isometric and respects $X^\perp$. Next we extend Y everywhere by imposing our two algebraic conditions $Y \in X^\perp$ and $< Y, Y >= 1$, (but obviously not the differential condition $[X, Y] = 0$).

Now, we return to the proof of the proposition, and let ∇ be the Levi-Civita connection of the Lorentz structure. Write $< X, X >= 0$, so that $0 = X < X, X >= 2 < \nabla_X X, X >$. Hence $\nabla_X X \in X^\perp$. In the same way $\nabla_Y X \in X^\perp$, and by the above fact, we also have $\nabla_X Y \in X^\perp$. Now $< X, Y >= 0$, implies :

$$\begin{aligned} 0 = Y < X, Y > &= < \nabla_Y X, Y > + < X, \nabla_Y Y > \\ &= < \nabla_X Y, Y > + < X, \nabla_Y Y >, \end{aligned}$$

since $[X, Y] = 0$. But $< Y, Y >= 1$, so $0 = X < Y, Y >= 2 < \nabla_X Y, Y >$. We thus obtain $< X, \nabla_Y Y >= 0$, that is $\nabla_Y Y \in X^\perp$. Therefore $X^\perp$ is invariant by ∇. This is equivalent to say that the leaves of $\mathcal{F}$ are geodesic.

Before investigating the structure of the asymptotic foliation, let us (locally) "normalize" further our adapted basis, in a manner very helpful for the calculations throughout this paper (an additional normalization will appear in 14.3).

Proposition 6.3. *For an adapted basis* $\{X, Y, Z\}$, *in a neighbourhood of a given point, we can assume that* $[X, Y] = [X, Z] = 0$, *and further that* $\nabla_Y Y = 0$.

Proof. As above, for the two first differential relations, we just take a transversal τ to X and define Y and Z on τ in any fashion such that they satisfy the algebraic constraints $< X, Y >= 0, < Y, Y >= 1, < Y, Z >= 0 =< Z, Z >$ and $< X, Z >= 1$. Next we define Y and Z in a neighbourhood of τ, by taking their images by ϕ^t. They still satisfy the algebraic relations as ϕ^t is an isometric flow, and therefore by definition $[X, Y] = [X, Z] = 0$. If we want further to have the relation $\nabla_Y Y = 0$ (locally), then we choose τ to be a ruled surface. That is τ is the union of geodesics in the leaves of $\mathcal{F}$, and Y is tangent to these geodesics. More precisely, we consider a curve $c(z), z \in [0, 1]$ transverse to $\mathcal{F}$, and an unitary vector $u(z)$ tangent at $c(z)$ to the leaf $\mathcal{F}_{c(z)}$. Then τ is the union of pieces of the geodesics determined by $u(z), z \in [0, 1]$, and Y is the tangent vector field to these geodesics. Therefore $\nabla_Y Y = 0$ in τ, and the same is true for the above extension of Y in a neighbourhood of τ.

Consider now the trace of the Lorentz structure in the tangent bundle of a leaf. This is a field of positive, but degenerate quadratic forms. Such an object is sometimes called a sub-riemannian structure, if it has a constant index of nullity. This notion may be helpfull as (for example) it permits to quantify the notion of (transversally) riemannian flow in the following way.

Definition 6.4. A 1-dimensional foliation $\mathcal{D}$ is riemannian if there is a sub-riemannian structure (in the supporting manifold) with nullity space the tangent space $T\mathcal{D}$ which is invariant by any parametrization of $\mathcal{D}$.

Therefore the foliation determined by the orbits of ϕ^t is leafwise riemannian. One important property of riemannian foliations is that they are, after passing to a suitable Steifel bundle, transversaly parallelisable [14]. This means in our (2−dimensional) case :

Proposition 6.5. *Let Ψ^t be the flow of the vector field Y. Then Ψ^t sends orbits of X into orbits of X (without preserving the parameter).*

Proof. This may be seen geometrically by considering in a given leaf, the semi length structure determined by $<>$. Observe that if x, y, x', y' are points in the leaf, x and y are in the same ϕ^t-orbit, and c and c' are curves joining x and x' to y and y' respectively, then length $(c) =$ length (c') if and only if x' and y' are in the same ϕ^t-orbit. This follows from our above definition which states that any reparametrization of ϕ^t preserves $< >$. Now apply this to the orbits of Ψ^t, which are in fact parametrized by arc-length : $\text{length}\{\Psi^t(x), 0 \leq t \leq T\} = T$.

Proposition 6.6. *A leaf of $\mathcal{F}$ is homeomorphic to a plane $(\mathbf{R}^2)$, a cylinder $(\mathbf{R} \times S^1)$ or a torus $(S^1 \times S^1)$.*

Proof. This follows from the classification by E. Ghys [5] of the codimension-one geodesic foliations of complete riemannian manifolds. But one may prove this in an elementary way for dimension 2. Take a leaf F that we assume to be simply connected, and denote by $\mathcal{D}$ the foliation determined by ϕ. Observe that an orbit of Ψ cuts all the leaves of $\mathcal{D}$. Indeed the subsets $\bigcup\{\Psi^t(D), t \in \mathbf{R}\}$, for D a leaf of $\mathcal{D}$, give a partition of F into open sets, and hence must be trivial. Therefore Ψ^t acts transitively on the quotient space $F/\mathcal{D}$, which must then be homeomorphic to $\mathbf{R}$. Let now Γ be a group of homeomorphisms of F preserving ϕ^t and Ψ^t. It acts by translation on $F/\mathcal{D}$, and so we get a homomorphism $h : \Gamma \to \mathbf{R}$. If h is injective, then Γ is abelian. In general, $\ker(h)$ contains elements which fix individually each leaf of $\mathcal{D}$. Since they commute with Ψ^t, they acts by translations on any fixed leaf D. Therefore $\ker(h)$ is abelian. In any case Γ is not free with more than one generator.

Next for a leaf which is not simply connected, we apply the above discussion to its universal cover, and deduce that its fundamental group is not free unless it is cyclic. Hence if open, this leaf must be a plane or a cylinder. On the other hand a compact leaf is a torus because it supports a non-singular vector field.

7. Partial ergodicity

Proposition 7.1. *Let $f : M \to \mathbf{R}$ be a smooth ϕ^t-invariant function. Then f is leafwise constant, i.e., f is constant in each leaf.*

Proof. If not, the following open invariant set in nonempty:

$$\mathcal{U} = \{x \in M \mid \ker d_x f \text{ does not contain } X^\perp(x)\}.$$

Therefore for $x \in \mathcal{U}, \ker d_x f$ is a lorentzian plane. In particular the second isotropic direction (other than that of X) determines a 1−dimensional invariant sub-bundle along $\mathcal{U}$. To finish the proof, we use the following lemma, which follows immediatly from the derivative cocycle formula (5.1).

Lemma 7.2. *Let $E \to \mathcal{U}$ be an invariant continuous 1−sub-bundle along an invariant subset $\mathcal{U}$. Then for any recurrent point x in $\mathcal{U}$, $E(x)$ coincides with the direction of $X(x)$, i.e., $E(x) = \mathbf{R} \cdot X(x)$.*

8. Properties of the connection along the leaves

We study now the Levi-Civita connection restricted to the leaves.

Lemma 8.1. *We have $\nabla_X X = 0$, that is, the orbits $t \to \phi^t(x)$ are affinely parametrized geodesics.*

Proof. Recall that the Killing property of X is expressed by the following infinitesimal condition : for any x, the covariant derivative map $u \in T_x M \to \nabla_u X \in T_x M$ is skew symmetric. In particular for any vector field $T, < \nabla_X X, T > + < X, \nabla_T X >= 0$. But $< X, \nabla_T X >= (T < X, X >)/2 = 0$, since $< X, X >= 0$. Therefore $< \nabla_X X, T >= 0$ for any T, that is $\nabla_X X = 0$.

Lemma 8.2. *There is a ϕ^t-invariant function $a : M \to \mathbf{R}$, such that $\nabla_Y X = aX$.*

Proof. We first note that $\nabla_Y X$ does not depend on the choice of Y. That is if Y' is another unitary vector field tangent to $X^\perp$, with the same orientation as Y, then $\nabla_{Y'} X = \nabla_Y X$. Indeed such Y' has the form $Y' = Y + fX$ for some function f. This implies by the preceding lemma that $\nabla_Y X = \nabla_{Y'} X$. It follows that this vector field is ϕ^t-invariant. Moreover, we may assume (6.2) that $[X, Y] = 0$ and hence $< \nabla_Y X, Y >=< \nabla_X Y, Y >= (X < Y, Y >)/2 = 0$. Therefore $\nabla_Y X = aX$, for some function a, necessarily invariant.

Proposition 8.3. *Let R be the curvature tensor of $(M, <>)$. Then $R(X, Y)X = 0$ and $R(X, Y)Y = \gamma X$ for some ϕ^t-invariant function γ, which is thus leafwise constant by partial ergodicity* (7.1). *Moreover, $\gamma = -a^2$.*

Proof. Assume $[X, Y] = 0$. Then $R(X,Y)X = \nabla_X \nabla_Y X - \nabla_Y \nabla_X X = \nabla_X(aX) = 0$, because a is ϕ^t-invariant. This proves the first formula. For the second, as in the proof of the above lemma we note that $R(X,Y)Y$ does not depend upon the choice of Y. Indeed for $Y' = Y + fX$, from the first equality $R(X,Y)fX = 0$ we get $R(X, Y + fX)(Y + fX) = R(X,Y)Y$. Next, observe that $< R(X,Y)Y, Y >= 0$, and hence $R(X,Y)Y = \gamma X$ for some ϕ^t-invariant function γ. Now by 6.3, we may assume $\nabla_Y Y = 0$, and by partial ergodicity $Y(a) = 0$, since a is ϕ^t-invariant. A direct calculation gives $\gamma = -a^2$.

Proposition 8.4. *The restriction of the connection ∇ on any leaf of $\mathcal{F}$ is locally symmetric.*

Proof. Fix a leaf F, and continue to denote the restriction of the connection and the curvature on it by ∇ and R. By partial ergodicity, $\nabla_Y X = aX$ and $R(X,Y)Y = \gamma X$, where a and γ are some constants. Let ∇R be the covariant derivative of R. Then

$$\begin{aligned}\nabla R(A,B,C,D) =& \nabla_A(R(B,C)D) - R(\nabla_A B, C)D \\ & - R(B, \nabla_A C)D - R(B,C)\nabla_A D.\end{aligned}$$

Each of the vectors A, B, C, D will be X or Y. We assume $0 = [X, Y]$, and recall our formulas : $\nabla_X X = 0, \nabla_Y X = aX, R(X,Y)X = 0$ and $R(X,Y)Y = \gamma X$.

Fact 8.5. *If 2 elements from $\{B, C, D\}$ equal X, then $\nabla R(A, B, C, D) = 0$.*

Proof. Indeed in this case all the derivatives $\nabla_A B, \nabla_A C$ and $\nabla_A D$ are proportional to X. Hence each term is proportional to $R(X,Y)X$ and so vanishes.

Fact 8.6. *If $A = X$ and one element from $\{B, C, D\}$ also equals X, then $\nabla R(A, B, C, D) = 0$.*

Proof. Indeed in this case $\nabla R(X, B, C, D) = \nabla_X R(B, C)D - R(B, C)\nabla_X D$. In every case $R(B,C)D$ is collinear with X, and so $\nabla_X R(B,C)D = 0$. If $D = Y$ then B or C equals X. But $\nabla_X D = aX$ and therefore $R(B,C)\nabla_X D$ is a multiple of $R(X,Y)X$ which is 0.

Now the remaining cases are when 3 elements from $\{A, B, C, D\}$ equal Y.

$$\begin{aligned}1)\quad \nabla R(X,Y,Y,Y) =& \nabla_X(R(Y,Y)Y) - R(\nabla_X Y, Y)Y \\ & - R(Y, \nabla_X Y)Y - R(Y,Y)\nabla_X Y \\ =& 0 - R(aX, Y)Y - R(Y, aX)Y - 0 \\ =& 0.\end{aligned}$$

By 6.3, we may assume $\nabla_Y Y = 0$.

$$\begin{aligned} 2)\ \nabla R(Y,X,Y,Y) =& \nabla_Y R(X,Y)Y - R(\nabla_Y X, Y)Y \\ & - R(X, \nabla_Y Y)Y - R(X,Y)\nabla_Y X \\ =& \nabla_Y \gamma X - R(aX,Y)Y - 0 - R(X,Y)aX \\ =& a\gamma X - \gamma a X = 0. \end{aligned}$$

The same for $\nabla R(Y,Y,X,Y)$.

$$\begin{aligned} 3)\ \nabla R(Y,Y,Y,X) =& \nabla_Y R(Y,Y)X \\ & - R(\nabla_Y Y, Y)X - R(Y, \nabla_Y Y)X - R(Y,Y)\nabla_Y X \\ =& 0 - 0 - 0 - 0 = 0. \end{aligned}$$

9. Symmetric connections in dimension 2

Let AG be the group of orientation preserving affine transformations of the real line. It is generated by the homotheties $\{g^t\}, g^t(x) = e^t x$, and translations $\{h^t\}, h^t x = x + t$. We have $g^{-t} h^s g^t = h^{se^{-t}}$. Its Lie algebra is generated by the two corresponding infinitesimal generators G and H, satisfying $[G,H] = -H$. As any Lie group, AG has a canonical bi-invariant, torsion free, complete and locally symmetric connection. It is defined in the Lie algebra level by : $\nabla_u v = \frac{1}{2}[u,v]$. Its curvature tensor is given by $R(u,v)v = \frac{1}{4}[v,[u,v]]$. Therefore we get in the case of AG : $R(G,H)H = 0$ and $R(H,G)G = \frac{-1}{4}H$. This looks like the situation of our leaves when putting $H = X$ and $G = Y$, at least for $\gamma = -a^2 = \frac{-1}{4}$. Our goal in this section is in fact to prove that in general, if $a \neq 0$, the leaves are locally affinely isomorphic to AG. For this, note first that only the sign of γ has sense. The exact value of γ deals with the sub-riemannian metric, or when fixing the vector field Y; if we change Y (or G) by a multiple, we can rescale γ to -1.

To understand the structure of AG, we represent it in the upper half-plane (like the 2-hyperbolic space) as follows : $g^t h^s \in AG \to (s, e^t) \in \mathbf{R} \times \mathbf{R}^+ = H^+$. The geodesics in AG are left (or right) translations of one-parameter groups (this is the case for any group). They are mapped in H^+ to (nonparametrized) straight lines. This means that the canonical connection ∇, and the flat connection ∇' (inherited from $\mathbf{R}^2$) are projectively equivalent.

Note that ∇' is also a bi-invariant connection on AG. This may be seen in the multiplication law of AG: any fixed left or right translation is affine.

The isometry group for ∇ is generated by left and right translations. It is then $AG \times AG$ acting on AG by $(f_1, f_2)f = f_1 f f_2^{-1}$.

One verifies on the other hand that the isometry group for ∇', i.e., the group of affine diffeomorphisms preserving the upper half plane H^+, is isomorphic to the direct product $AG \times AG$.

Remark 9.1. Note that in order for a connection to be bi-invariant, it suffices that it is invariant under $AG \times \{h^t, t \in \mathbf{R}\}$ or $\{h^t, t \in \mathbf{R}\} \times AG$. Furthermore the bi-invariant connection on AG are exactly the convex combinations $\alpha\nabla + (1-\alpha)\nabla'$ for $\alpha \in [0,1]$. All of them are projectively flat and satisfy the curvature formulas: $R(G,H)H = 0$ and $R(H,G)G = \gamma(\alpha)G$. But none of them other than ∇ is complete, and only ∇ and ∇' are locally symmetric.

Proposition 9.2. *Let F be a 2−manifold endowed with a locally symmetric connection ∇ and two nonsingular vector fields X and Y such that $R(Y,X)X = -R(X,Y)X = 0$ and $R(X,Y)Y = -a^2X$. with $a \neq 0$. Then F is locally isomorphic to AG with its canonical connection (observe that we do not assume that X and Y are Killing fields and that from the Remark above, the local symmetry property is necessary).*

Proof. Recall that for a locally symmetric connection any tangent vector u determines a local transvection flow along the geodesic that it determines. If $x(t)$ is this geodesic, then the transvection flow is defined by $T_u^t = S_{x(t/2)}S_{x(0)}$, where $S_{x(t)}$ is the symmetry around $x(t)$. Thus T_u^t is connection preserving, and preserves the geodesic determined by u, and DT_u^t equals the parallel transport along $x(t)$.

Observe that we have a well defined sub-riemannian structure $<,>$ by $R(X,u)u =< u,u > X$; its kernel is just the direction of X. It is preserved by affine isometries since this is so for the curvature. In particular, since any geodesic is the orbit of a transvection one-parameter group, this geodesic is everywhere lightlike whence it is somewhere lightlike. That is the orbits of X are geodesic. More precisely let A^t be the tranvection flow determined by some $X(x_0)$, and A the associated Killing field. Then A is collinear with X, that is the orbits of A are exactly (all) the lightlike geodesics. Indeed these geodesics may be defined as sets of points with some fixed semi-distance from the orbit of $X(x_0)$. Therefore they are (individually) preserved by A^t.

Let $y(t)$ be a geodesic with $y(0) = x_0$ and $< y'(0), y'(0) >= 1$. Let B be the vector field extending this geodesic defined by taking its images by A^t, that is, $[A,B] = 0$. The orbits of B are unitary geodesics since A is a Killing field.

We now show that all the covariant derivatives determined by A and B are uniquely derived from our conditions. Therefore the analogous construction for AG yields an affine isomorphism; in other words, with respect to the coordinnates system defined by A an B, the covariant derivatives laws are the same as for AG.

First of all, recall that on locally symmetric spaces, isometry-invariant tensors are parallel (that is because there are enough transvections flows inducing parallel transport along given geodesics). This means for our sub-riemannian metric that we have the usual derivation formula: $C < D, E >=< \nabla_C D, E > + < C, \nabla_C E >$.

Let us prove that $\nabla_A A = 0$. Since the orbits of A are geodesic, we have $\nabla_A A = bA$ for some function b, which is invariant by A^t since this flow preserves both A and the connection ∇.

Since $< B, B >= 1$, we have $< \nabla_A B, B >= 0$ and therefore $\nabla_A B = \nabla_B A = cA$ for some function c which, for the same reasons as above, is A^t-invariant.

We have

$$\begin{aligned} 0 = R(B, A)A &= \nabla_B \nabla_A A - \nabla_A \nabla_B A = \nabla_B(bA) - \nabla_A(cA) \\ &= B(b)A + bcA - cbA = B(b)A. \end{aligned}$$

Thus $B(b) = 0$, that is, b is B-invariant. But $b = 0$ along the A-orbit of x_0 by the definition of transvection flows. It then follows that b equals 0 everywhere, that is $\nabla_A A = 0$.

We consider now

$$R(A, B)B = \nabla_A \nabla_B B + \nabla_B \nabla_A B = -a^2 A.$$

We know by the construction of B that $\nabla_B B = 0$, so that $\nabla_B(cA) = B(c)A + c^2 A = -a^2 A$, that is $B(c) + c^2 = -a^2$. As in the case of b, initially (i.e., along the A-orbit of x_0) c equals 0. Therefore c is well defined from the differential equation $B(c)+c^2 = -a^2$. To finish, observe that we may rescale the constant a to be the same as for AG by just replacing Y by some constant multiple αY.

Proposition 9.3. *The leaves of $\mathcal{F}$ are complete.*

Proof. Let F be a leaf of $\mathcal{F}$. It is modeled on AG for $a > 0$ and on $\mathbf{R}^2$ if $a = 0$. The proof for the two cases is the same. Let us, to fix notation, consider the case of AG. To simplify notation let us suppose that F is simply connected. Thus we have a developping map $d : F \to AG$. Any connection preserving flow on F is the pull-back of such a flow on AG. In particular, we have

Fact 9.4. *The restriction of ϕ^t to F projects to a flow k^t on AG. By the curvature formula (in F and AG), the infinitesimal generator K is collinear with H. So the expression of k^t is $k^t : x \to h^{bt} x h^{ct}$, for some* *constants b and c. In particular the orbits of K (like those of H) in the upper half plane model are horizontal lines.*

Denote by $\mathcal{D}$ and $\mathcal{D}'$ the foliation in F and AG defined respectively by the orbits of ϕ^t and k^t (or h^t). From the above fact, we deduce that for any orbit $D \in \mathcal{D}$, the developping map is a homeomorphism from a strip (i.e., a connected $\mathcal{D}-$invariant subset) containing D to an analogous one around $d(D)$.

Fix D_0 and let D_t be the leaf at distance t from D_0, in the sense of the sub-riemannian metric, and on a given side, say the positive one determined by the orientation. Such a leaf exists since the flow Ψ^t (of the vector field Y) is complete in the compact manifold M. Let $y_0 \in D_0$ and $u \in T_{y_0}F$ a unit vector so that $< u, u >= 1$. It determines a geodesic $y(t)$ defined in a maximal interval $[0, \epsilon[$. Note that $y(t) \in D_t$, and hence in order to prove that this geodesic is complete, i.e., $\epsilon = +\infty$, it suffices to show that it cuts all the leaves D_t, for $t \geq 0$. But if not $y(t)$ should accumulate to D_ϵ, (note that by the fact that $y(t)$ belongs to D_t, our geodesic cuts each D_t at most one time).

Now apply the developping map d to get a (nonhorizontal) half-line which is contained in a strip around $d(D_\epsilon)$; impossible.

10. Compact leaves and "differentiable ergodicity"

Proposition 10.1. *The foliation $\mathcal{F}$ has no compact leaves.*

Proof. We prove that if F is a toral leaf, then the restriction of ϕ^t to F is equicontinuous. Indeed, $F = AG/\Gamma$, where Γ is a subgroup of $AG \times AG$ isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ and acting freely, dicontinuously and uniformly on AG. Moreover Γ must centralise the one-parameter group $k^t = (h^{bt}, h^{ct})$ (see 9.4). Denote by p_1 and p_2 the projections $AG \times AG \rightarrow AG$ onto the first and the second factors, respectively. We recall that in AG, the centralizer of $\{h^t\}$ is exactly $\{h^t\}$. Hence if both b and c are non zero, Γ should be contained in $\{(h^t, h^s), (t, s) \in \mathbf{R}^2)\}$. But this group preserves the foliation $\mathcal{D}'$ of AG (notation from the proof of 9.4), and acts nonuniformly. Therefore we may assume for example $b = 0$, and $p_1(\Gamma)$ to be nontrivial in AG. As an abelian subgroup $p_1(\Gamma)$ is contained in a one-parameter group $(l^t, 1)$. Thus this l^t centralizes Γ and so defines a flow Ψ^t on $F = AG/\Gamma$, transverse to ϕ^t. Futhermore this two flows in F commute, since this is the case for the one-parameter groups $(l^t, 1)$ and $(1, h^{ct})$.

To get a contradiction, we just apply 7.2. Therefore $\mathcal{F}$ has no compact leaves.

Corollary 10.2. *Let $f : M \rightarrow \mathbf{R}$ be a smooth ϕ^t-invariant func-*

tion. Then f is constant (by analogy with the usual notion of ergodicity, we call this property the differentiable ergodicity of ϕ^t; see for instance [17]).

Proof. By 7.1, f is constant along the leaves of $\mathcal{F}$. Hence, if non-constant, a generic level of f contains a compact leaf. But this does not exist by the proposition above.

11. Invariant quadratic forms

Let $q : TM \times TM \to \mathbf{R}$ be a symmetric bilinear form. With the help of an adapted basis $\{X, Y, Z\}$ and identification with $(R^{2,1}, <>), q$ is determined by a map $x \in M \to A_x$, where A_x is a 3×3 matrix, symmetric with respect to $<>$ (i.e., $< u, A_x(v) > = < A_x(u), v >$). Thus $q_x(u, v) = < u, A_x v >$.

Proposition 11.1. *The set of symmetric bilinear ϕ^t-invariant forms are exactly those given by constant matrices of the form :*

$$A = \begin{pmatrix} \lambda & 0 & \alpha \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ and α are constants.

Proof. The invariance condition for q, means $(c(t, x))^{-1} A_x c(t, x) = A_{\phi^t(x)}$, where $c(t, x)$ is the derivative cocycle. Let

$$B^t = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $B^t = \exp tb$, and $c(t, x) = \exp T(t, x)b$, where $T(t, x)$ is a real additive cocycle tending to ∞ when $t \to \pm\infty$ (§5).

Let B^t act by conjugation in the space $\mathcal{M}_{3\times 3}$ of 3×3 matrices.

Claim. A recurrent matrix A for this action is a fixed point. That is A commutes with B^t.

Proof. Since b is nilpotent, we have $B^t = 1 + tb + \frac{t^2}{2} b^2$. Therefore $B^{-t} A B^t = (1 - tb + \frac{t^2}{2} b^2) A (1 + tb + \frac{t^2}{2} b^2)$. If this polynomial takes values near A, for t large, then all its nonconstant coefficients must vanish. This means that A commutes with b, and so with B^t.

Now by continuity of the map $x \to A_x$, if x is recurrent for the dynamical system (M, ϕ^t), then A_x is recurrent for the adjoint action of B^t. Therefore A_x commutes with $c(t, x)$ and $A_x = A_{\phi^t(x)}$. We deduce from the density of recurrent points that the map $x \to A_x$ is ϕ^t-invariant, not only as a 2-tensor, but also as a matrix valued function. Thus by differentiable ergodicity A_x equals a constant matrix A.

Next by eigenspaces consideration, we prove, as A commutes with B^t that A has the form

$$A = \begin{pmatrix} \lambda & \beta & \alpha \\ 0 & \lambda & \gamma \\ 0 & 0 & \lambda \end{pmatrix}.$$

By computation, we find $\beta = -\gamma$. Moreover the symmetry property of A, more precisely $< A(e_2), e_3 >=< e_2, A(e_3) >$, gives $\beta = \gamma$. Hence $\beta = \gamma = 0$ and so A has the promissed form.

To finish the proof, we note that conversely any such matrix determines a ϕ^t-invariant symmetric bilinear form. For example if $\lambda = 1$, we obtain an invariant scalar product which gives the same values as the old one, in all the cases, but $< Z, Z >= \alpha$, instead of 0.

12. The Ricci curvature

The Ricci tensor *Ricc* of M is a ϕ^t-invariant quadratic form. Thus by the previous section, it may be expressed by means of a constant matrix $rI + \delta J$, where we set :

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We would be happy if it is trivial in the sense that $\delta = 0$, i.e., $Ricc = r <>$. That is M is an Einstein manifold. Indeed in dimension 3 this implies that M is of constant sectional curvature. “Unfortunately” in general δ may be nontrivial. Our method is then to equip M with a family of metrics $<>_\alpha$, obtained by matrices $I + \alpha J$, and so to consider $\delta(\alpha)$ as a function of α. Our hope is to get an Einstein metric for some α.

Recall now [15] that the Levi-Civita connection associated to a pseudo-riemannian metric $<>$ is given by the following :

$$\begin{aligned} 2 < \nabla_V W, T > =& V < W, T > + W < T, V > - T < V, W > \\ & - < V, [W, T] > + < W, [T, V] > + < T, [V, W] >, \end{aligned}$$

where V, W and T are three vector fields.

A one-parameter family of metrics. Consider now the metrics $< u, v >_\alpha =< u, v + \alpha J(v) >$.

Fact 12.1. *All the scalar products of the basis vectors do not depend on* α, *except* $< Z, Z >_\alpha$ *which equals* α. *More generally, we have* $< A, B >_\alpha = < A, B >_0$, *once* A *or* B *is a combination of* X *and* Y *only (i.e., tangent to* $\mathcal{F}$*). Conversely if two vector fields* A *and* A' *are such that for some* $\alpha \neq 0$ *and any vector field* B, $< A, B >_\alpha = < A', B >_0$, *then* A *equals* A' *and is tangent to* $\mathcal{F}$.

For any fixed α, all the scalar products of our basis are constant in M. Therefore we have

Fact 12.2. *Let* ∇^α *be the Levi-Civita connection of* $<>_\alpha$. *For vector fields* V, W, T, *each of which is an element of the basis* $\{X, Y, Z\}$, *we have*

$$2 < \nabla^\alpha_V W, T >_\alpha = - < V, [W, T] >_\alpha + < W, [T, V] >_\alpha + < T, [V, W] >_\alpha . \tag{1}$$

Corollary 12.3. *The connections* ∇^α *coincides with* $\nabla = \nabla^0$ *on the leaves of* $\mathcal{F}$, *that is,* $\nabla^\alpha_V W = \nabla_V W$, *whenever* V *and* W *are* X *or* Y.

Proof. Indeed, in (1), only T may depend on Z, and so in every case by 12.1, the right-hand side of (1) does not depend on α.

Notation. We consider now an adapted basis (in the sense of $<>$) satisfying the conditions of 6.3 in a neighbourhood of some given point. The only nontrivial bracket is $[Y, Z]$. Let us write it as

$$[Y, Z] = lX + mY + nZ,$$

where l, m and n are some functions.

Recall from 8.2 that for some constant a,

$$\nabla_Y X = \nabla_X Y = aX. \tag{2}$$

Fact 12.4. *We have*

$$n = -2a. \tag{3}$$

Moreover, $m = 0$ *and hence*

$$[Y, X] = lX - 2aZ. \tag{4}$$

Proof. Applying (1) to $V = X, W = Y$ and $T = Z$ gives

$$\begin{aligned} 2 < \nabla_X Y, Z > &= - < X, [Y, Z] > \\ &= - < X, lX + mY + nZ > \\ &= -n. \end{aligned}$$

By (2) we get $n = -2a$.

Now apply (1) to $V = W = Y$ and $T = Z$. Since $\nabla_Y Y = 0$, we get

$$\begin{aligned} 0 &= -< Y, [Y, Z] > + < Y, [Z, Y] > \\ &= -2 < Y, [Y, Z] > = -2 < Y, lX + mY + nZ > \\ &= -2m. \end{aligned}$$

Therefore $m = 0$.

Fact 12.5.

(5) $$\nabla^\alpha_X Z = -aY$$

Proof. Indeed, if we apply (1) to $V = X, Z = W$ and $T = X$ or $T = Z$, then we obtain 0 on the right-hand side. Therefore $\nabla^\alpha_X Z$ is a multiple of $Y : \nabla_X Z = bY$. Applying (1) to $T = Y$ yields

$$\begin{aligned} 2b &= 2 < \nabla^\alpha_X Z, Y >_\alpha = - < X, [Z, Y] >_\alpha \\ &= < X, [Y, Z] >_\alpha = n, \end{aligned}$$

which together with (3) implies that $b = -a$.

Fact 12.6.

(6) $$\nabla^\alpha_Z Z = (l - 2a\alpha)Y.$$

Proof. By (1) we immediately get $< \nabla^\alpha_Z Z, Z >_\alpha = < \nabla^\alpha_Z Z, X >_\alpha = 0$, and so $\nabla^\alpha_Z Z = b(\alpha)Y$. Again from (1) it follows that

$$\begin{aligned} 2b(\alpha) = < \nabla^\alpha_Z Z, Y >_\alpha &= - < Z, [Z, Y] >_\alpha + < Z, [Y, Z] >_\alpha \\ &= 2 < Z, [Y, Z] >_\alpha = 2 < Z, lX - 2aZ >_\alpha \\ &= 2(l - 2a\alpha). \end{aligned}$$

Fact 12.7.

(7) $$\nabla^\alpha_Y Z = a\alpha X - aZ.$$

Proof.

$$\begin{aligned} 2 < \nabla^\alpha_Y Z, Y >_\alpha &= - < Y, [Z, Y] >_\alpha + < Y, [Y, Z] > \\ &= 2 < Y, [Y, Z] > = 0, \quad \text{by (4)}. \end{aligned}$$

It then follows that $\nabla^\alpha_Y Z = b(\alpha)X + c(\alpha)Z$, so that

$$\begin{aligned} 2c(\alpha) = 2 < \nabla^\alpha_Y Z, X > &= < X, [Y, Z] > \\ &= < X, lX - 2aZ >= -2a, \end{aligned}$$

by (4). Hence $c(\alpha) = -a$. Now

$$2 < \nabla^\alpha_Y Z, Z >_\alpha =< Z, [Z, Y] > + < Z, [Y, Z] >= 0.$$

Thus

$$0 = 2 < b(\alpha)X - aZ, Z >_\alpha = 2(b(\alpha) - a\alpha) = 0,$$

which implies (7).

Fact 12.8.

$$\nabla^\alpha_Z Y = \nabla^\alpha_Y Z + [Z, Y] = (a\alpha - l)X + aZ \tag{8}$$

Curvature formulas. Let R^α be the curvature tensor of $<>_\alpha$.

Fact 12.9. *Since $\nabla^\alpha = \nabla$ on the leaves of $\mathcal{F}$, we have :*

$$\begin{aligned} R^\alpha(X, Y)X &= R(X, Y)X = 0, \\ R^\alpha(X, Y)Y &= R(X, Y)Y = -a^2 X, \\ < R^\alpha(X, Y)Y, Z >_\alpha &= < R^\alpha(X, Y)Y, Z >= -a^2, \\ < R^\alpha(Z, Y)Y, X > &= -a^2. \end{aligned}$$

Fact 12.10. $R^\alpha(Y, Z)X = (Y(l) + 4al - 5a^2\alpha)Y$.

Proof.

$$\begin{aligned} R^\alpha(Y, Z)Z =& \nabla^\alpha_Y \nabla^\alpha_Z Z - \nabla^\alpha_Z \nabla^\alpha_Y Z - \nabla^\alpha_{[Y,Z]} Z \\ =& \nabla^\alpha_Y((l - 2a\alpha)Y) - (\nabla^\alpha_Z(a\alpha X - aZ)) - \nabla_{lX-2aZ} Z \\ =& Y(l - 2a\alpha)Y + (l - 2a\alpha)\nabla^\alpha_Y Y - a\alpha(\nabla^\alpha_Z X) \\ &+ a\alpha\nabla^\alpha_Z Z - l\nabla^\alpha_X Z + 2a\nabla^\alpha_Z Z \\ =& Y(l)Y + 0 + a^2\alpha Y + a(l - 2a\alpha)Y + laY \\ &+ 2a(l - 2a\alpha)Y \\ =& (Y(l) + 4al - 5a^2\alpha)Y. \end{aligned}$$

Fact 12.11. $R^\alpha(X, Z)X = -2a^2 X$.

Proof. $R^\alpha(X, Z)X = \nabla_X \nabla_Z X - \nabla_Z \nabla_X X = \nabla_X(-aY) = -a^2 X$.

Fact 12.12. $R^\alpha(X, Z)Z = -a^2\alpha X + b(\alpha)Y + c(\alpha)Z$.

Proof. Indeed,

$$0 =< R^\alpha(X, Z)Z, Z >_\alpha = a(\alpha) + \alpha c(\alpha),$$

and

$$c(\alpha) = < R^\alpha(X,Z)Z, X >_\alpha \;=\; - < R^\alpha(X,Z)X, Z > \\ = - < -a^2X, Z > = a^2.$$

Hence $a(\alpha) = -a^2\alpha$.

Let now Ricc^α be the Ricci tensor for the metric $<>_\alpha$; that is, $\mathrm{Ricc}^\alpha(u,u)$ equals the trace of the linear map $A^\alpha_u : v \to R^\alpha(v,u)u$.

Fact 12.13. *$Ricc^\alpha(Z,Z) = (Y(l) + 4al) - 6a^2\alpha$.*

Proof. We have $A^\alpha_Z(Z) = 0$, and the coordinate of $A^\alpha_Z(Y)$ with respect to Y equals $< A^\alpha_Z(Y), Y >$, and therefore $< R^\alpha(Y,Z)Z, Y >_\alpha = (Y(l)+4al) - 5a^2\alpha$ by 12.10. Moreover 12.11 implies that the coordinate of $A^\alpha_Z(X)$ relative to X is $-a^2\alpha$. Thus $\mathrm{trace}(A^\alpha_Z) = (Y(l) + 4al) - 6a^2\alpha$.

Fact 12.4. *$Ricc^\alpha(Y,Y) = -a^2\alpha$.*

Proof. We show that the coordinate of $A^\alpha_Y(X)$(resp.$A^\alpha_Y(Z)$) relative to X(resp.Z) is $-a^2$. For this we use the relations $< R^\alpha(X,Y)Y, X >_\alpha = 0$ and $< R^\alpha(X,Y)Y, Z > = < R^\alpha(Z,Y)Y, X >_\alpha = -a^2$.

Combining all these calculations and 11.1 yields

Proposition 12.15. *The Ricci curvature of the metric $<>_\alpha$ is obtained (from $<>$ and not $<>_\alpha$) by the matrix $-2a^2I + \delta(\alpha)J$, where $\delta(\alpha) = -6a^2\alpha + \delta_0$, and δ_0 is a constant.*

13. The case $\mathbf{a} \neq 0$

When $a \neq 0$, there is α such that $\delta(\alpha) = -6a^2\alpha + \delta_0 = -2a^2\alpha$ (since $6 \neq 2$); that is, $-2a^2I + \delta(\alpha)J = -2a^2(I + \alpha J)$, and so $\mathrm{Ricc}^\alpha = -2a^2 <>_\alpha$ Thus in dimension 3, $(M, <>_\alpha)$ has constant negative (sectionnal) curvature $\frac{-a^2}{2}$. Now to simplify the notation let us suppose that $\alpha = 0$, so that our initial metric $<>$ has constant curvature $\frac{-a^2}{2}$. It is known that after rescaling (multiplicative) constants, $(M, <>)$ is locally isometric to $PSL(2, \mathbf{R})$, endowed with its Killing form.

Denote by N the universal cover of $PSL(2,\mathbf{R})$ as a Lorentz manifold and by G its universal cover as a group. The isometry group of N is generated by the left and the right translations which commute, and hence this group is $G \times G$. Therefore our manifold M has a $(G \times G, N)$ structure. Let $d : \tilde{M} \to N$ be the developing map, and $Hol : \pi_1(M) \to G \times G$ be the holonomy. The lifting $\tilde{\phi}^t$ of ϕ^t in $\tilde{M}$ is induced by an isometric flow A^t, with an infinitesimal generator A. This Killing field is, like $\tilde{X}$, lightike in $d(N)$, and hence everywhere by analyticity. Write $A = (A_1, A_2) \in \mathcal{G} \times \mathcal{G}$. Obviously A commutes with each A_i, and also with the holonomy group $Hol(\pi_1(M))$. If each A_i is nontrivial, then we get two nontrivial one-parameter groups in M, which commute with ϕ^t, being impossible by 7.2 since we assume ϕ^t nonequicontinuous. Therefore, we may assume for example that $A = (0, A_2)$, i.e., A^t acts by the right translations on N. From the lightlike character of A, we deduce that $ad(A_2)$ is nilpotent and that its orthogonal space is exactly its normalizer algebra, which generates a subgroup $H \subset G$, isomorphic to AG.

In fact A^t is a parabolic subgroup, so its projection in $SL(2, \mathbf{R})$ is conjugate to a one-parameter unipotent group $\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbf{R}\}$. Let $\mathcal{H}$ be the asymptotic foliation of A, that is the left invariant foliation $\mathcal{H}_x = x \cdot H$. Note that the geometric structure of M is in fact modeled on $(G \times \mathbf{R}, N)$, where the $\mathbf{R}$ factor acts as the right translations by A^t,

Completeness. Of course d maps $\tilde{\mathcal{F}}$ into $\mathcal{H}$. We have already proved (§9)) that the leaves of $\mathcal{F}$ are geodesically complete. Therefore d maps homeomorphically leaves of $\tilde{\mathcal{F}}$ into leaves of $\mathcal{H}$. Furthermore, for a small transversal (at some given point) τ to $\tilde{\mathcal{F}}$, its $\tilde{\mathcal{F}}-$ saturation Ω is homeomorphically mapped by d onto its image. Therefore the completeness problem is reduced to a $1-$dimensional one. More precisely, we see that in order for d to be a homeomorphism onto its image, it is enough to show that $\tilde{\mathcal{F}}$ admits a global transversal, that is, a transversal curve cutting every leaf (this is equivalent to the fact that $\tilde{\mathcal{M}}/\tilde{\mathcal{F}}$ is Hausdorff and hence homeomorphic to $\mathbf{R}$).

Observe now that for the foliation $\mathcal{H}$, any timelike geodesic is a global transversal (warning : this can never be the case for lightlike or spacelike geodesics). We first verify this by an example, in the geometric situation of $PSL(2, \mathbf{R})$ instead of its universal cover N. So $PSL(2, \mathbf{R})$ may be identified with the unit tangent bundle of the $2-$hyperbolic space $\mathbf{H}^2$, and $\mathcal{H}$ with the weak horocycle foliation. Then the fibers of $T^1\mathbf{H}^2 \to \mathbf{H}^2$, are timelike geodesics which cut exactly once any leaf of $\mathcal{H}$ (they represent in fact the circle at infinity of $\mathbf{H}^2$, which is the space of $\mathcal{H}-$leaves). Observe now that this remains true in the universal cover. To handle now the case of general timelike (nonparametrised) geodesics, remark that they are all obtained as images of the previous ones, by isometries preserving $\mathcal{H}$.

Let $c(t), t$ in a maximal interval $]\epsilon^-, \epsilon^+[$, be a timelike geodesic in $\tilde{M}$, and Ω its $\tilde{\mathcal{F}}-$saturation. Then d is a homeomorphism from Ω onto $d(\Omega)$. If Ω is not all of $\tilde{M}$, then at least one side of our geodesic (say when $t \mapsto \epsilon^+$) approaches a boundary leaf F_0, without cutting it. That is, for any saturated one-sided neighbourhood V of $F_0, c(t) \in V$ for $t \geq t_V$. In particular $c(t)$ escapes every compact subset of V. But for

V suitably small, this picture translates in the model N, since d is a homeomorphism on V. We get a timelike half geodesic which escapes compact sets and stays in a small neighbourhood of a leaf of $\mathcal{H}$. But the so corresponding complete geodesic can not hit that leaf. This contradicts the previous discussion, and therefore d is a homeomorphism onto its image.

Thus M would be geodesically complete if $d(\tilde{M}) = N$. But if not, $d(\tilde{M})$ will have one or two leaf boundary components. Therefore, after, if necessary, passing to a double cover of M, we may assume that the holonomy group preserves each boundary component. Assume one of these leaves is $\mathcal{H}_1$, the leaf of the identity element 1. Then its isotropy group in $G \times \mathbf{R}$ is exactly $H \times \mathbf{R}$ (and H is isomorphic to AG). But $H \times \mathbf{R}$ acts freely in $N \setminus \mathcal{H}_1$. Therefore the holonomy group $Hol(\pi_1(M))$ should be a lattice in $H \times \mathbf{R}$. But this is impossible since H is not unimolular, and the proof is complete.

14. The case $\mathbf{a} = 0$

In this section, we suppose $\mathbf{a} = 0$. Our goal is to prove that non-equicontinuous lightlike Killing fields do not exist in this case (for compact manifolds).

Fact 14.1. *X is a parallel vector field, that is, $\nabla_A X = 0$ for any vector field A.*

Proof. This follows from the formulas of §12, which we summarize in the following:

Fact 14.2. *We have*

$$\nabla_X X = \nabla_Y Y = [X, Y] = [X, Z] = \nabla_Y X = \nabla_Z Y = \nabla_Y Z = 0,$$
$$\nabla_Z Z = [Y, Z] = lX.$$

Furthermore the derivative $Y(l)$ equals a constant c.

Proof. The first two equalities are obtained from §12, by putting $a = 0$, and the rest follows from proposition 12.15, which states that the Ricci tensor is expressed by the matrix $Y(l)J$, and therefore $Y(l)$ equals a constant c.

In riemannian geometry, the de Rham decomposition theorem shows that any parallel vector field gives rise to a local riemannian product. Thus, the above relations must imply the space being flat (i.e., in the fact above, $c = 0$). This does not extend to Lorentzian geometry, as examples satisying all the relations for $c \neq 0$ exist (see below). Therefore our nonexistence proof (of nonequicontinuous lightlike Killing fields) will be achieved by global considerations. But to begin, let us investigate the local structure of these manifolds (without attempting to give a systematic classification which seems to be possible).

Fact 14.3. *In a neighbourhood of a given point x_0, we can normalize our adapted basis, such that other than the relations of 6.3, the $Z-$ orbit of x_0 is any given lightlike geodesic containing x_0 and transverse to $\mathcal{F}$. In this case the function l is constant on the leaves of the foliation $\mathcal{L}$ generated by X and Z (it exists because $[X, Z] = 0$). Furthermore the leaf $\mathcal{L}_{x_0}$ is geodesic and l vanishes in it.*

Proof. The first part is an improvement of 6.3. So in the proof of that Proposition, we choose the curve $c(z)$, to be a prescribed lightlike geodesic, such that $Z(c(z)) = \frac{\partial c}{\partial z}$ and $< Z(x_0), X(x_0) >= 1$. This implies that $< Z(c(z)), X(c(z)) >= 1$ for any $z \in [0,1]$ (the Noether first integral) since X is Killing and $c(z)$ is geodesic. Next we define Y along $c(z)$ to be orthogonal to Z, tangent to $\mathcal{F}$ and unitary, and so the rest of the proof works like 6.3. Thus we have $\nabla_Z Z = 0$, on the leaf $\mathcal{L}_{x_0}$. From 14.2 it follows that this leaf is geodesic. Again by 14.2 we obtain $\nabla_Z Z = lX$ and hence l vanishes in $\mathcal{L}_{x_0}$. Now $[Y, Z](l) = lX(l) = 0$ since X preserves all the data. Writing $[Y, Z]l = YZ(l) - ZY(l) = 0$, and putting $Y(l) =$ constant yield $YZ(l) = 0$. This means $Z(l)$ is $Y-$invariant. But l and so $Z(l)$ vanish in $\mathcal{L}_{x_0}$. Thus $Z(l) = 0$ everywhere (in a neighbourhood of x_0). In particular l is constant on the leaves of $\mathcal{L}$.

A Lie algebra. Write $T = [Y, Z] = lX$. Then $\{X, Y, Z, T\}$ generate a Lie algebra $\mathcal{G}$ with relations :$[X, Y] = [X, Z] = [X, T] = 0, [Y, Z] = T, [Z, T] = [Z, lX] = Z(l)X + l[Z, X] = 0$ and $[Y, T] = cX$, where $c = Y(l)$.

This algebra is solvable and determines a simply connected Lie group G. The isotropy group of x_0 is the one-parameter group I generated by T. The quotient G/I is diffeomorphic to $\mathbf{R}^3$, and our initial fields X, Y, Z, T correspond to the left invariant vectors fields on G. Observe now that for A an element of the Lie algebra generated by X and Z, also the right invariant field on G that A determines, projects on G/I. This follows from the fact that A centralises I. Let f^t be the flow determined by A in G/I; that is, $f^t(xI) = xA^tI$, where $A^t = \exp tA$ is the one-parameter group determined by A. Then f^t commutes with the flow of each vector field of X, Y, Z and T, since they are left invariant.

But then f^t also preserves the metric since it is determined by the scalar products among the fields X, Y and Z.

Note now that in $\mathcal{L}_{x_0}$, the left and the right actions of the elements of the Lie algebra generated by X and Z are the same. In particular the right action is transitive on $\mathcal{L}_{x_0}$. This proves:

Fact 14.4. *Given x_0, X, Y and Z as above, the group of isometries preserving X, Y and Z acts transitively on the leaf $\mathcal{L}_{x_0}$.*

Next, to get more isometries, consider all the possible Z, or in other words all the lightlike geodesics at x_0. The union of all the so obtained geodesic leaves $\mathcal{L}_{x_0}$ contains an open set containing x_0 in its boundary. By the previous fact, all these points are the images of x_0 by isometries preserving X. By varying x_0, we get:

Proposition 14.5. *The Lie algebra $\mathcal{G}$ of local Killing fields preserving X (and so also $\mathcal{F}$) acts locally transitively on M.*

Notation. Denote by $\mathcal{I}$ and $\mathcal{H}$ the isotropy algebra of x_0 and the leaf $\mathcal{F}_{x_0}$ respectively.

Fact 14.6. *$\mathcal{H}$ is an ideal of $\mathcal{G}$.*

Proof. First note that like X, the $2-$ plane field $X^\perp$ (or equivalently $\mathcal{F}$), is parallel. That is for any vector fields A and B, if B is tangent to $X^\perp$, then $\nabla_A B$ is tangent to $X^\perp$. Indeed $< B, X >= 0$, and hence $< \nabla_A B, X >= - < B, \nabla_A X >= 0$, since X is parallel. Next observe that if furthermore A is a local Killing field commuting with X, then $< \nabla_B A, X > + < B, \nabla_X A >= 0$, since A is Killing. But $\nabla_X A = \nabla_A X = 0$. Hence $\nabla_B A$ is tangent to $X^\perp$. This implies that $[A, B]$ is tangent to $X^\perp$, and so $\mathcal{H}$ is an ideal of $\mathcal{G}$.

The dimension of $\mathcal{H}$ may be 2 or 3. The first case is "trivial" as this will be seen later.

Proposition 14.7. *If $\dim \mathcal{H} = 3$, then it is isomorphic to the Lie algebra of the Heisenberg group with X corresponding to the center.*

Proof. The universal cover of a leaf of $\mathcal{F}$ is identified to $\mathbf{R}^2$, with the orbits of X corresponding to the parallels to the $x-$axis and the sub-riemannian metric corresponding to dy^2 (see §9). Thus X is a linear combination of the vector fields determined by a transvection flow $(x, y) \to (x + ty, y)$ and a translation flow along the $x-$axis.

But since X is nonsingular, it must correspond to a pure translation flow. Therefore the centralizer of X in the group of affine transformations preserving dy^2 is generated by X together with A (the transvection field above) and B, the translation flow along the $y-$axis (see §9). We have $[A, B] = X$, and hence the so generated Lie algebra is of Heisenberg type.

We deduce from this that $\mathcal{G}$ is solvable. Let G be the corresponding simply connected Lie group. Let x_0 be a given point. Its isotropy algebra, if nontrivial, corresponds to a transvection field A in the leaf $\mathcal{F}_{x_0}$. It generates in G a one-parameter group I, closed in G since G is solvable and simply connected [2]. Therefore the quotient space G/I exists and M is modeled on it. Let $\Gamma \subset G$ be the holonomy group.

It acts by the left translations on G/I. We are now going to find a contradiction with nonequicontinuity of the flow of X.

Case 0. The isotropy group I is trivial (or equivalently $dim\mathcal{H} = 2$). Equip G with a left invariant riemannian metric. It passes to M since Γ acts by the left translations. It is also preserved by X since it is central. Therefore X is a Killing field for a riemannian metric and in particular equicontinuous.

Now, we assume that I is nontrivial. Then the isotropy algebra $\mathcal{I}$ is generated by a transvection flow A, and the isotropy algebra $\mathcal{H}$ of the leaf $\mathcal{F}_{x_0}$ is generated by X, A and B, a translation flow along the y-axis. Let H be the (Heisenberg) group determined by $\mathcal{H}$.

Case 1. $\triangle = \Gamma \cap H$ is nontrivial. We know that the leaves of $\mathcal{F}$ are complete. Hence the leaf $\mathcal{F}_{x_0}$ is homeomorphic to the quotient of H/I by $\triangle$ which acts freely on H/I. Since $\triangle$ is nontrivial, this leaf is a cylinder and so $\triangle$ is cyclic. We first show that $\triangle$ intersects trivially the one-parameter group generated by X. Indeed, if not the flow X on M should be periodic (not only for x_0) since X is central. Therefore $\triangle$ is contained in a one-parameter group different from that of X and A, and is invariant by Γ. This determines in M a nontrivial vector field commuting with X. This is impossible by 7.2.

Case 2. $\Gamma \cap H = \{1\}$. Thus Γ injects in $G/H \approx \mathbf{R}$ and therefore is abelian. Let a be an element of Γ (and hence not belonging to H) which is not central in G. Let L be a one-parameter group containing a (this is not necessarily unique).

Claim. Γ centralises L (this yields as above a nontrivial vector field in M, commuting with X. Contradiction).

Proof. Let Z be the centralizer of a in G. It contains L and Γ since Γ is abelian. It may be written as a semi-direct product $Z = LN$, where N is the centralizer of a in the Heisenberg group H. Therefore N is a closed connected Lie subgroup of H; the connectedness follows from the fact that the exponential map in H is injective, and so if an element belongs to N, then all the one-parameter group containing it is lying in N. The dimension of N may be 0, 1 or 2, but not 3 since a is not central in G.

i) $\dim N = 0$. This implies $\Gamma \subset L$, and the claim is obvious.

ii) $\dim N = 1$. The one-parameter group L acts on N as a one-parameter group g^t of the exterior automorphisms, with the element g^1 corresponding to a, which is trivial (= identity) since by definition a centralizes N. But in dimension 1, this implies g^t is trivial and so $Z = LN$ is abelian. Therefore the claim is true in this case.

iii) dim$N = 2$. The subgroup N must contain the center of H (i.e., the one-parameter group determined by X) since it is known that the center of the Heisenberg group has no supplementary subgroup. Therefore at the Lie algebra (of N) level, the one-parameter group g^t has an eigenvector. But, g^1 trivial means g^t is conjugate to the a one-parameter group of rotations. This is impossible unless g^t is trivial. Therefore Z and in particular Γ centralizes L. This proves the claim in this case and so finishes the nonexistence proof of nonequicontinuous Killing fields in the case $a = 0$.

Examples of spaces satisfying $\mathbf{a} = 0$. Consider in $\mathbf{R}^3$, the following three vector fields X, Y and Z:

$$\begin{aligned} X &= (1,0,0), \\ Z &= (0,0,1), \\ Y &= (cyz + dy + ez, 1, 0). \end{aligned}$$

Then $[X,Z] = [X,Y] = 0$ and $[Y,Z] = (cy+e)X$. Thus $Y(cy+e) = \frac{\partial}{\partial y}(cy+e) = c$. We construct a Lorenztian metric on $\mathbf{R}^3$ by setting $< X,X >=< X,Y >=< Y,Z >=< Z,Z >= 0$ and $< Y,Y >=< X,Z >= 1$. That is the metric $dxdz + dy^2 - (cyz + dy + ez)dydz$.

This metric has a Ricci tensor $R(u,u) = c < u, Ju >$ where J is as in §12, the linear map that sends X and Y to 0 and sends Z to X.

It seems that in the representation above only the parameter c is relevant, and all the spaces with $a = 0$ are of this form. Another way to justify this hypothesis is to consider the homogenous space G/I as above, where G is a normal extension of the Heisenberg group H. Such a space possesses a $G-$invariant Lorentz metric if $Ad(I)$ preserves a Lorentz scalar product on $\mathcal{G}/\mathcal{I}$. The space of isomorphism classes of these extensions has (as the parameter c above) dimension 1.

15. Singularities

In this section, we prove that singularities can not occur for lightlike nonequicontinuous vector fields. Suppose the contrary and let S be the singular locus:

$$S = \{x \in M / X(x) = 0\} = \{x \in M / \phi^t x = x, \forall t\}.$$

As the set of fixed points of isometries, S is the union of a finite disjoint set of closed geodesic submanifolds. It can not contain a $2-$dimensional (i.e., of codimension 1) submanifold, since otherwise ϕ^t would be equicontinuous by 3.6. Hence S is a finite disjoint union of points and closed geodesics.

Fact 15.1. *The foliation $\mathcal{F}$ (defined in $M - S$ by the 2−plane field $X^{\perp}$) extends continuously to M. Furthermore the foliation by orbits of X extends continuously to M, as a 1−dimensional geodesic foliation $\mathcal{D}$.*

Proof. Let $x_0 \in S$, and V be a compact convex neighbourhood of it. This is not a flow box for $\mathcal{F}$, since $B - S$ is not compact; but as the leaves are geodesic, the foliation in it is topologically trivial. Let D_0 be the connected component of x_0 in S. Then $D_0 = \{x_0\}$ or D_0 is a closed geodesic containing x_0. The situation here is geometricaly similar to $\mathbf{R}^3 - \{(0,0,0)\}$ or $\mathbf{R}^3 - \mathbf{R} \times \{(0,0)\}$ foliated by planes. For $x \in B$, let P_x be its plaque in B, that is the connected component of x in $\mathcal{F}_x \cap B$, and let $\bar{P}_x$ be its closure in B.

Case 1. $D_0 = \{x_0\}$. We shall see that this is impossible. Indeed in this case only one plaque P_0 contains x_0 in its closure, and also the orbits of X define a foliation by geodesics in $P_0 = \bar{P}_0 - \{x_0\}$. But by 8.1, the orbits of X are in fact parametrized geodesics, and so exit every small convex subset. Thus these pieces of geodesics are closed in $\bar{P}_0$, and none of them contains x_0 in its closure. Contradiction.

Case 2. D_0 is a geodesic. As in the above argument, we show that the foliation by orbits of X, in $B - D_0$, extends continuously to B, such that D_0 becomes a leaf. In particular D_0 is lightlike. Again as above, we see that if for a plaque $P_0, \bar{P}_0 \cap D_0 \neq \emptyset$, then $D_0 \subset \bar{P}_0$. Moreover $\bar{P}_0$ is necessarily the degenerate geodesic half-surface defined by the lightlike geodesic D_0. It then follows that if P_0 and P_1 are distinct plaques containing D_0, then $\bar{P}_0 \cup \bar{P}_1$ is a "smooth" geodesic surface. This is exactly that defined by the orthogonal of D_0. This finishes the proof of the continuous extensions for $\mathcal{F}$ and $\mathcal{D}$.

Now the proofs in the previous sections, for the nonsingular case, extend (continuously!) to the singular one. In particular there is a constant a, the same for all the leaves, which affinely classify them. We may assume that a equals 1 or 0. That is the leaves are modeled on AG or $\mathbf{R}^2$. Let us restrict ourself to AG since it requires more care.

Let x_0 be a singular point, and x_1 be a nonsingular point in its leaf. Near x_1, the leaf-connection depends continuously on the leaf. Therefore, in a neighbourhood B of x_1 we may find developping maps $d_x : P_x \to AG$, from the plaques P_x to AG, which depend continuously on x. Furthermore the flow of x corresponds to a one-parameter group $\{(h^{\alpha(x)t}, h^{\beta(x)t})\}$ in $AG \times AG$, with α and β continuous in x. It acts on AG by $z \to h^{\alpha(x)t} z h^{\beta(x)t}$ (see 9.4). Fortunately, there is a local way to detect if such a one-parameter group is somewhere singular:

Fact 15.2. *The flow $z \to h^{\alpha t} z h^{\beta t}$ on AG is nonsingular if and only if α and β have the same sign and at least one of them does not vanish.*

Proof. An element z of AG has the form $z = g^u h^s$. Thus $h^{\alpha t} g^u h^s h^{\beta t} = g^u h^{\alpha t e^{-u}} h^s h^{\beta t}$. Therefore z is singular if and only if $\alpha e^{-u} + \beta = 0$.

Now since there are at most a finite set of leaves of $\mathcal{F}$ containing singularities, we get: $\alpha(x)$ and $\beta(x)$ have the same sign in a dense set of B. The same is true for x_1. Hence $\alpha(x_1) = \beta(x_1) = 0$, since we know that the leaf $\mathcal{F}_{x_1}$ contains x_0 as a singularity. But this implies all the points of the leaf are singular. Contradiction.

16. Proofs of Theorem 2 and Theorem 3.

Proof of Theorem 3. In the Anosov case, this was proved by E. Ghys, who also remarked that this is true for small deformations of any general flow. In particular for a deformation of a surface group $\Gamma \subset PSL(2,\mathbf{R})$ in $PSL(2,\mathbf{R}) \times \{h^t\}$ by $\gamma \longrightarrow (\gamma, h^{c(\gamma)})$ and c small (for a fixed generating set), the so obtained flow is smoothy orbit equivalent to the horocyclic flow. In order to prove the theorem in the general case, we just remark as W. Goldmann [6] did that any deformation is equivalent to a small one. For this we conjugate Γ in $PSL(2,\mathbf{R}) \times PSL(2,\mathbf{R})$ by an element $(1, g^t) = l^t$; thus $l^t \Gamma' l^{-t} = \{(\gamma, g^t h^{c(t)} g^{-t})\}$, since $g^t h^s g^{-t} = h^{se^{-t}}$, the deformation is small for t large enough.

Proof of Theorem 2.

1) *The suspension case.* Let A be a hyperbolic linear automorphism on a torus $\mathbf{T}^2$. We consider the Lorentz structure defined by $< u^s, u^u >= \omega(u^s, u^u)$, where u^s and u^u are vectors tangent to the stable and the unstable directions respectively, and ω is a linear volume form. This metric is flat, as it lifts to a Minkowski metric on $\mathbf{R}^2$. Furthermore, A preserves it, and so the product (flat) metric on $\mathbf{T}^2 \times [0,1]$ passes to a flat metric on $\mathbf{T}^2 \times [0,1]/_{(x,1)\sim(Ax,0)}$, and the suspension flow $\partial/\partial t$ preserves it. One may also multiply it by different positive constants along the factors $\mathbf{T}^2$ and $[0,1]$, to obtain another flat metric. To prove conversely that any invariant metric is of this type, observe that by the Anosov property, the stable and the unstable directions must be orthogonal to the flow. Therefore, the metric on $\mathbf{T}^2 \times [0,1]$ is a product since ϕ^t is isometric. The problem is thus reduced to check that up to a constant, the above metric on $\mathbf{T}^2$ is the only one which is invariant by A. This follows from the ergodicity.

2) Now, let us show that a Ghys flow canonically determines (up to a constant) a metric of constant negative curvature.

2.1) *The Anosov case.* Consider the one-form ω such that $\omega(X) = 1$

and $Ker(\omega)$ equals the sum of the stable and the unstable spaces. Consider now the Kanai metric $<,>$ such that $Ker(\omega)$ is orthogonal to X, $< X, X >= 1$, and $< Y, Z >= d\omega(Y, Z)$ for $Y,\ Z \in Ker(\omega)$. Of course this is defined by means of the flow only. We know that this is nondegenerate since $Ker(\omega)$ is not integrable. Next, a calculation that we omit since it is straightforward and very elemantary compared to that of §12, shows that this metric has a constant negative curvature.

Remark. The consideration of the Kanai metric and the calculation mentioned above gives, as any one expects, an alternative simple proof of the Ghys classification of Anosov flows with smooth stable and unstable distributions. Of course this only works when $Ker(\omega)$ is nonintegrable, but the alternative case is easily seen to correspond to suspensions.

2.2) *The parabolic case.* We start with a metric $<,>$ of constant curvature. Thus up to a constant any other invariant metric has the form $<,>_\alpha$ described in §12, and by 12.15, the Ricci curvature of $<,>_\alpha$ is given by the matrix $-2a^2I - 6a^2\alpha J$. Hence $<,>_\alpha$ is of Einstein type if and only if $\alpha = 0$. Therefore, there is (up to a constant) exactly one invariant metric of constant negative curvature.

3) The proofs of most of the points in this part are straightforward. Let us just mention how to prove that all the metrics $<,>_\alpha$ for $\alpha > 0$, are locally isometric to $<,>_1$. More precisely, the corresponding metrics on $PSL(2, \mathbf{R})$ are (globally) isometric (analogously, for $\alpha < 0$, the metrics are isometric to $<,>_{-1}$). For this, recall that the Lie algebra $\mathcal{G}$ of $PSL(2, \mathbf{R})$ is generated by $X,\ Y,\ Z$ with relations $[Y, X] = -X,\ [Y, Z] = Z$ and $[X, Z] = Y$ (warning : this notation for X and Y is opposite to the usual one, but of course compatible with our earlier notation). Up to a factor 2, the Killing form is given by $< Y, Y >= 1,\ < X, X >=< Z, Z >= 0$ and $< X, Z >= -1$.

The scalar product $<,>_\alpha$, invariant by $exp(ad_X)$, takes the same values as the Killing form except $< Z, Z >_\alpha = \alpha$.

Let h_α be the left invariant metric on $PSL(2, \mathbf{R})$ determined by $<,>_\alpha$. We claim that the right translation $x \longrightarrow xg^t$ by the hyperbolic one-parameter group g^t maps h_1 isometrically to h_α if $\alpha = e^{-2t}$. Indeed, for this, it suffices to show that $Ad(g^t)$ maps the scalar product $<,>_1$ to $<,>_\alpha$, and this is obvious, since $Ad(g^t)Z = e^tZ,\ Ad(g^t)X = e^{-t}X$ and $Ad(g^t)Y = Y$.

4) This follows from the discussion above.

Remark 16.1. It is seen from [8] that all the present metrics are complete. So, compact Lorentz 3-manifolds admitting nonequicontinuous Killing fields are complete.

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